The Weinberg-Witten result for the limits on massless particles: a closer look at the proof.

Liza Rozenberg

The Weinberg-Witten theorem outlines a constraint on possible values of helicity [1] for massless particles with specified conserved quantities. This bears importance in the context of investigating allowable particle theories of gravity since it is a well-known fact that any quantum theory of gravity must have a massless spin 2 particle [3]. Additionally, the theorem allows us to test the validity of existing quantum theories by checking if their predictions about spin conflict with existence of various massless particles in our world.

The theorem consits of two parts:

Theorem 1 *(i) A Lorentz-covariant conserved four-vector current J^μ forbids existence of massless particles with spin $j > \frac{1}{2}$ with charge $Q = \int d^3x J^0 \neq 0$.*

(ii) A Lorentz-covariant conserved energy-momentum tensor $T^{\mu\nu}$ forbids existence of massless particles with spin $j > 1$ with energy-momentum four-vector $P^\mu = \int d^3x T^{0\mu} \neq 0$.

The original proof in [5] was done using *delta*-function normalization of one-particle states. In this paper, we will examine the proof of the theorem in a more mathematically rigorous fashion using a physical definition of one-particle states.

Before proceding with the proof, we should first resolve some confusion that comes up in the terminology of the theorem statement. In this paper and in the original paper [5], the term *Lorentz covariance* is used; in other sources, including [3], the term *Poincaré covariance* is used instead. This is not a contradiction: the Lorentz group is a subgroup of the Poincaré group, so the former covariance condition implies the latter. Formally, the Poincaré group is the semidirect product (a generalization of direct product) of the Lorentz group with the group of space-time translations. Space-time covariance is implicit in any theory we want to use, so all that remains to check is covariance with respect to the Lorentz group. Hence, most sources use the terms *Lorentz covariance* and *Poincaré covariance* interchangeably.

1 Proof of the theorem

The strategy of the proof is to reach a contradiction for the value of the matrix elements

$$\langle p', \pm j | J^\mu | p, \pm j \rangle \qquad \text{or} \qquad \langle p', \pm j | T^{\mu\nu} | p, \pm j \rangle. \tag{1}$$

The first step is to show that based on Lorentz covariance, the matrix elements (1) do not vanish in the limit $p' \to p$. The second step is showing using Lorentz transformation properties that (1) do have to vanish in this limit for spins $j > \frac{1}{2}$ and $j > 1$ respectively.

[1] We sometimes use the term spin in analogy with the group theoretic terminology, although we recognize that spin is ususally used for massive particles in physics settings.

1.1 Step 1

The non-vanishing of these matrix elements can be shown using the assumptions that $Q \neq 0$ and $P^\mu \neq 0$ in the statement of Theorem 1. The proof in the original paper [5] claims that Lorentz covariance forces the limiting behavior $p \to p'$ of these matrix elements to be

$$\begin{aligned} \langle p', \pm j | J^\mu | p, \pm j \rangle &\to \frac{q p^\mu}{E(2\pi)^3} \\ \langle p', \pm j | T^{\mu\nu} | p, \pm j \rangle &\to \frac{p^\mu p^\nu}{E(2\pi)^3} \end{aligned} \tag{2}$$

Continuity of the these matrix elements at $p' = p$ is not necessary for the argument to go through – this becomes clear when we define our states not using the usual delta-function normalization $\langle p', \pm j | p, \pm j \rangle = \delta^{(3)}(p - p')$, but a sharply-peaked normalization which approaches the delta-function in some parameter limit. Let us see how this works.

Recall that in [5], charge and momentum are defined by relying on measurement – we measure nearly forward scattering caused by exchange of spacelike but nearly lightlike massless vector bosons or gravitons. In mathematical terms, this constitutes taking the limit $p - p' \to 0$. DIrectly using equality $p = p'$ is not a physically accurate condition, and since the rest of the argument relies on physical quantities, we should check that we can avoid using this unphysical set-up.

Let us consider bulding the states based on the smeared normalization using a bump function. It is an example of a more general class of functions known as mollifiers. For our purposes, a bump function with parameter a is a function which is sharply peaked at the center of a compact interval which in the limit $a \to 0$ gives the usual delta function. To sumarize, we want the bump function $f(x)$ to obey the following key properties [2]:

(i) $f(x)$ is smooth and compactly supported on $\mathbb{R}^n$

(ii) $\int_{\mathbb{R}^n} f(x) dx = 1$

(iii) $\lim_{a\to 0} f_a(x) = \lim_{a \to 0} \frac{1}{a^n} f(\frac{x}{a}) = \delta^{(n)}(x)$

Such functions are also allowed to obey other properties which we are free to impose on them, as long as the above three are obeyed.

A good example of a bump function is

$$f(x) = \begin{cases} N e^{-\frac{1}{1-x^2}} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases} \tag{3}$$

where N denotes the appropriate normalization which makes the integral of f over the entire space equal to 1. From now on, we will denote the bump function used in our state definitions by $\delta_a = \frac{1}{a} f(\frac{x}{a})$, where f is some appropriate mollifier and the variable x is n-dimensional.

We define the charge and four-momentum of massless one-particle states as follows:

$$|p, \pm j\rangle_{ph} = \int d^3 \vec{p}' \delta_a^{(3)}(\mathbf{p}' - \mathbf{p}) |p', \pm j\rangle \tag{4}$$

We require the following normalization, which defines additional properties of $\delta_a^{(3)}$ via the last equality in (5) [3]:

$$\begin{aligned} {}_{ph}\langle p_1, \pm j | p_2, \pm j \rangle_{ph} &= \int d^3 q_1 d^3 q_2 \delta_a^{(3)}(\mathbf{q}_1 - \mathbf{p}_1) \delta_a^{(3)}(\mathbf{q}_2 - \mathbf{p}_2) \delta^{(3)}(\mathbf{q}_1 - \mathbf{q}_2) \\ &= \int d^3 q_1 \delta_a^{(3)}(\mathbf{q}_1 - \mathbf{p}_1) \delta_a^{(3)}(\mathbf{q}_1 - \mathbf{p}_2) \\ &= \delta_a^{(3)}(\mathbf{p}_1 - \mathbf{p}_2) \end{aligned} \tag{5}$$

[2] A detailed review of mollifiers and their properties can be found in [1].

[3] We can check that this property still leaves the three mollifier properties of δ_a intact.

In all our discussions, we will use the boldface notation for spatial vectors.

We should note also that that the eigenmomentum is now the superposition of the form

$$P^{\mu}|p,\pm j\rangle_{ph} = \int d^3p'\delta_a^{(3)}(\mathbf{p'}-\mathbf{p})p'^{\mu}|p',\pm j\rangle \tag{6}$$

Examining this result on physical grounds, we should identify the momentum eigenvalue with p^{μ} because (6) is sharply peaked at this momentum. Since we imposed that the bump function reduces to the usual delta function, we can show that

$$\langle p',\pm j|p,\pm j\rangle = \lim_{a\to 0}\delta_a^{(3)}(\vec{p}-\vec{p}') \tag{7}$$

This allows us to write

$$\begin{aligned} {}_{ph}\langle p',\pm j|Q|p,\pm j\rangle_{ph} &= q\delta_a^{(3)}(\mathbf{p}-\mathbf{p'}) \\ {}_{ph}\langle p',\pm j|P^{\mu}|p,\pm j\rangle_{ph} &= p^{\mu}\delta_a^{(3)}(\mathbf{p}-\mathbf{p'}) \end{aligned} \tag{8}$$

Expressions for these matrix elements can alternatively be obtained by integration over large but finite volume of space (due to our finite normalization) of energy-momentum four-vector or tensor. Throughout the derivation, we use the notation that the (spatial) momentum operator $\hat{\mathbf{P}}$ acts on the eigenstate $|p,\pm j\rangle$ to give $\mathbf{p}|p,\pm j\rangle$.

$$\begin{aligned} {}_{ph}\langle p',\pm j|Q|p,\pm j\rangle_{ph} &= \int_{V_a} d^3\mathbf{x}\,{}_{ph}\langle p',\pm j|J^0(t,\mathbf{x})|p,\pm j\rangle_{ph} = \int_{V_a} d^3\mathbf{x}\,{}_{ph}\langle p',\pm j|e^{i\hat{\mathbf{P}}\cdot\mathbf{x}}J^0(t,\mathbf{0})e^{-i\hat{\mathbf{P}}\cdot\mathbf{x}}|p,\pm j\rangle_{ph} \\ &= \int_{V_a} d^3\mathbf{x}\int d^3\tilde{p}'\delta_a^{(3)}(\tilde{\mathbf{p}}'-\mathbf{p'})\int d^3\tilde{p}\delta_a^{(3)}(\tilde{\mathbf{p}}-\mathbf{p})\langle\tilde{p}',\pm j|e^{i\hat{\mathbf{P}}\cdot\mathbf{x}}J^0(t,\mathbf{0})e^{-i\hat{\mathbf{P}}\cdot\mathbf{x}}|\tilde{p},\pm j\rangle \\ &= \int_{V_a} d^3\mathbf{x}\int d^3\tilde{p}'\int d^3\tilde{p}\delta_a^{(3)}(\tilde{\mathbf{p}}'-\mathbf{p'})\delta_a^{(3)}(\tilde{\mathbf{p}}-\mathbf{p})e^{i(\tilde{\mathbf{p}}'-\tilde{\mathbf{p}})\cdot\mathbf{x}}\langle\tilde{p}',\pm j|J^0(t,\mathbf{0})|\tilde{p},\pm j\rangle \end{aligned} \tag{9}$$

Note that δ_a scales as $\frac{1}{a^3}$. To proceed, we need to use a saddle point approximation which will give us a quantity whose correction vanishes in the limit $a\to 0$. We are allowed to use saddle point approximation because the mollifier decays faster than a Gaussian function, for which a saddle point approximation is appropriate.

For smooth function $g(x)$ and bump function $\delta_a^{(3)}(x) = \frac{1}{a^3}f(\frac{x}{a})$ with a saddle point at x_0, we can write the following result [2]:

$$\int d^3x g(x)\delta_a^{(3)}(x) = \left(\frac{2\pi}{|f(x_0/a)''|}\right)^{\frac{3}{2}} g(x_0)f(x_0)(1+\mathcal{O}(a^3)) \tag{10}$$

Here, f is a smooth function which goes into the definition of a particular bump function. The saddles of the two bump functions in (9) are at $\tilde{\mathbf{p'}} = \mathbf{p'}$ and $\tilde{\mathbf{p}} = \mathbf{p}$ respectively. Performing the saddle point approximation twice on our expression in (9), we get:

$$\begin{aligned} {}_{ph}\langle p',\pm j|Q|p,\pm j\rangle_{ph} &= (2\pi)^3\int_{V_a} d^3\mathbf{x}e^{i(\mathbf{p}-\mathbf{p'})\cdot\mathbf{x}}(\langle p',\pm j|J^0(t,\mathbf{0})|p,\pm j\rangle + \mathcal{O}(a^3)) \\ &= (2\pi)^3\delta_a^{(3)}(\mathbf{p'}-\mathbf{p})\langle p',\pm j|J^0(t,0)|p,\pm j\rangle + \mathcal{O}(a^3) \end{aligned} \tag{11}$$

The second term in the final expression will vanish in the limit $a \to 0$. Likewise, we have

$$\begin{aligned}
{}_{ph}\langle p', \pm j|P^\mu|p, \pm j\rangle_{ph} &= \int_{V_a} d^3\mathbf{x}\, {}_{ph}\langle p', \pm j|T^{0\mu}(t, \mathbf{x})|p, \pm j\rangle_{ph} = \int_{V_a} d^3\mathbf{x}\, {}_{ph}\langle p', \pm j|e^{i\hat{\mathbf{P}}\cdot\mathbf{x}}T^{0\mu}(t, \mathbf{0})e^{-i\hat{\mathbf{P}}\cdot\mathbf{x}}|p, \pm j\rangle_{ph} \\
&= \int_{V_a} d^3\mathbf{x} \int d^3\tilde{p}'\delta_a^{(3)}(\tilde{\mathbf{p}}' - \mathbf{p}') \int d^3\tilde{p}\delta_a^{(3)}(\tilde{\mathbf{p}} - \mathbf{p})\langle \tilde{p}', \pm j|e^{i\hat{\mathbf{P}}\cdot\mathbf{x}}T^{0\mu}(t, \mathbf{0})e^{-i\hat{\mathbf{P}}\cdot\mathbf{x}}|\tilde{p}, \pm j\rangle \\
&= \int_{V_a} d^3\mathbf{x} \int d^3\tilde{p}' \int d^3\tilde{p}\delta_a^{(3)}(\tilde{\mathbf{p}}' - \mathbf{p}')\delta_a^{(3)}(\tilde{\mathbf{p}} - \mathbf{p})e^{i(\tilde{\mathbf{p}}' - \tilde{\mathbf{p}})\cdot\mathbf{x}}\langle \tilde{p}', \pm j|T^{0\mu}(t, \mathbf{0})|\tilde{p}, \pm j\rangle \\
&= (2\pi)^3 \int_{V_a} d^3\mathbf{x} e^{i(\mathbf{p} - \mathbf{p}')\cdot\mathbf{x}}(\langle p', \pm j|T^{0\mu}(t, \mathbf{0})|p, \pm j\rangle + \mathcal{O}(a^3)) \\
&= (2\pi)^3\delta_a^{(3)}(\mathbf{p}' - \mathbf{p})\langle p', \pm j|T^{0\mu}(t, 0)|p, \pm j\rangle + \mathcal{O}(a^3)
\end{aligned} \tag{12}$$

Once again, the second term in the final expression vanishes.

In both (9) and (12), we used the following limit to reach the final expression:

$$\lim_{a\to 0} \int_{V_a} d^3x e^{i(p'-p)\cdot x} = \delta^{(3)}(p' - p) = \lim_{a\to 0} \delta_a^{(3)}(p - p') \tag{13}$$

Now, comparing (8) with (9) and (12), we conclude that

$$\begin{aligned}
\lim_{p'-p\to 0} \langle p', \pm j|J^0(t, 0)|p, \pm j\rangle_{ph} &= \frac{q}{(2\pi)^3} \\
\lim_{p'-p\to 0} \langle p', \pm j|T^{0\mu}(t, 0)|p, \pm j\rangle_{ph} &= \frac{p^\mu}{(2\pi)^3}
\end{aligned} \tag{14}$$

The final step is to recall that J^μ and $T^{\mu\nu}$ are Lorentz-covariant objects. This means they transform under elements Λ of the Lorentz group [4]. We also need to recall that the boost in μ direction is given by $\Lambda^\mu_0 = \frac{p^\mu}{|p|} = \frac{p^\mu}{E}$. In total then, we obtain the transformations:

$$\begin{aligned}
J^\mu &= \Lambda^\mu_0 J^0 = \frac{p^\mu}{E} J^0 \\
T^{\mu\nu} &= \Lambda^\mu_0 T^{0\nu} = \frac{p^\mu}{E} T^{0\nu}
\end{aligned} \tag{15}$$

Using this in (14), we get

$$\begin{aligned}
\lim_{p'-p\to 0} \langle p', \pm j|J^\mu|p, \pm j\rangle_{ph} &= \frac{qp^\mu}{(2\pi)^3E} \neq 0 \\
\lim_{p'-p\to 0} \langle p', \pm j|T^{0,\mu}(t, 0)|p, \pm j\rangle_{ph} &= \frac{p^\mu p^\nu}{(2\pi)^3E} \neq 0
\end{aligned} \tag{16}$$

where in the final step of each line, the inequality follows from the assumption in Theorem 1 that the massless particles are charged under the appropriate currents.

Taking the limit $a \to 0$, the same results also hold for states without subscript ph, i.e. at the limit point $p' = p$. Using the more physical definition of the state normalization, we obtain a more general result which does not only hold for that limit point where the delta function does not vanish, but in an open neighborhood determined by a. If we didn't use this smeared normalization, we would not have been able obtain that result without assumption of continuity.

1.2 Step 2

Now, we have to show using a different approach that the matrix elements have to vanish for certain value of spin. We do this by first recalling that p, p' are light-like. Let's say their spatial components are at angular separation of α, which implies that

$$(p + p')^2 = 2(p \cdot p') = 2(\mathbf{p} \cdot \mathbf{p}' - |\mathbf{p}||\mathbf{p}'|) = 2|\mathbf{p}||\mathbf{p}'|(\cos\alpha - 1) \leq 0 \tag{17}$$

First, let us look at the case $\alpha \neq 0$. In this scenario, $p+p'$ is timelike, so by Lorentz covariance we may pick to work in a frame where $p+p'$ has zero spatial component [4]. So, let's take

$$p = (|\mathbf{p}|, \mathbf{p}), \qquad p' = (|\mathbf{p}|, -\mathbf{p}) \tag{18}$$

Let us rotate p, p' around the $\mathbf{p}$ direction by an angle θ. Note that rotating by $+\theta$ around $\mathbf{p}$ implies a rotation by $-\theta$ around $\mathbf{p'} = -\mathbf{p}$. From [4], we know that helicity representations of a Lorentz transformation Λ act on the massless state vector as

$$U(\Lambda, 0)|p, j\rangle = \sqrt{\frac{(\Lambda p)^0}{p^0}} e^{ij\theta(\Lambda, p)} |\Lambda p, j\rangle \tag{19}$$

where $\theta(\Lambda, p)$ is the angle defined by the rotation component of Λ. Hence, the one-particle states transform under rotation by θ as

$$\begin{aligned} |p, \pm j\rangle &\to e^{\pm i\theta j}|p, \pm j\rangle \\ |p', \pm j\rangle &\to e^{\mp i\theta j}|p', \pm j\rangle \end{aligned} \tag{20}$$

Note also that helicity (for massless particles) is invariant under Lorentz transformations [4].

Thus, the rotation results in the following transformation for the matrix elements:

$$\begin{aligned} \langle p', \pm j|J^\mu(t,0)|p, \pm j\rangle &\to e^{\pm 2i\theta j}\langle p', \pm j|J^\mu(t,0)|p, \pm j\rangle \\ \langle p', \pm j|T^{\mu\nu}(t,0)|p, \pm j\rangle &\to e^{\pm 2i\theta j}\langle p', \pm j|T^{\mu\nu}(t,0)|p, \pm j\rangle \end{aligned} \tag{21}$$

Similarly, recall that a rotation may also be descibed by the fundamental representation of the Lorentz group acting on $J^\mu, T^{\mu\nu}$ [3]. Since helicity is Lorentz-invariant, we can write the right-hand sides (RHS) of (21) using the fundamental representation $R(\theta)$:

$$\begin{aligned} e^{\pm 2i\theta j}\langle p', \pm j|J^\mu(t,0)|p, \pm j\rangle &= R(\theta)^\mu_\nu \langle p', \pm j|J^\nu(t,0)|p, \pm j\rangle \\ e^{\pm 2i\theta j}\langle p', \pm j|T^{\mu\nu}(t,0)|p, \pm j\rangle &= R(\theta)^\mu_\rho R(\theta)^\nu_\sigma \langle p', \pm j|T^{\rho\sigma}(t,0)|p, \pm j\rangle \end{aligned} \tag{22}$$

On the RHS of (22), we are transforming the currents, and on the LHS, we are instead tranforming the basis vectors. Since we want these to give the same result, we equate the two sides as in (22).

Now, since $R(\theta)$ is a rotation matrix, its eigenvalues must be roots of unity, namely $e^{+i\theta}, e^{-i\theta}$, or 1. Looking at the first result in (22), this implies that all components of the $J^\mu(t,0)$ matrix elements have to vanish except for $2j \in \{0,1\}$. Likewise, all matrix elements of $T^{\mu\nu}(t,0)$ must vanish unless $2j \in \{0,1,2\}$. Now, since helicity is Lorentz-invariant, and J^μ, $T^{\mu\nu}$ are Lorentz-covariant (and hence have their vanishing properties preserved throughout reference frame changes), the same conclusions follow for matrix elements of J^μ and $T^{\mu\nu}$ respectively.

Now, the case for $\alpha = 0$ does not correspond to the limit $p' \to p$. Instead, this corresponds to the case where $p + p'$ is lightlike. This is the limit of particles basically moving side by side along the same trajectory. This is not quite the limit we would like to consider since this is not the "nearly-forward scattering" that we need, but it is still instructive to study this case.

We may consider $\mathbf{p}, \mathbf{p'}$ parallel to each other, meaning the rotation results in the following single-particle state transformation:

$$\begin{aligned} |p, \pm j\rangle &\to e^{\pm i\theta j}|p, \pm j\rangle \\ |p', \pm j\rangle &\to e^{\pm i\theta j}|p', \pm j\rangle \end{aligned} \tag{23}$$

The matrix elements are now invariant under the transformation, since the p' state now rotates in the same way as the p state:

$$\begin{aligned} \langle p', \pm j|J^\mu(t,0)|p, \pm j\rangle &\to \langle p', \pm j|J^\mu(t,0)|p, \pm j\rangle \\ \langle p', \pm j|T^{\mu\nu}(t,0)|p, \pm j\rangle &\to \langle p', \pm j|T^{\mu\nu}(t,0)|p, \pm j\rangle \end{aligned} \tag{24}$$

Once again, looking at the Lorentz transformation in its fundamental representation, we have:

$$\begin{aligned}\langle p', \pm j|J^{\mu}(t,0)|p, \pm j\rangle &= R(\theta)^{\mu}_{\nu}\langle p', \pm j|J^{\nu}(t,0)|p, \pm j\rangle \\ \langle p', \pm j|T^{\mu\nu}(t,0)|p, \pm j\rangle &= R(\theta)^{\mu}_{\rho}R(\theta)^{\nu}_{\sigma}\langle p', \pm j|T^{\rho\sigma}(t,0)|p, \pm j\rangle\end{aligned} \tag{25}$$

The components of the matrix elements along $\mathbf{p}, \mathbf{p'}$, and time directions do not have to vanish since their rotation eigenvalues are 1 which satisfies (24) and (25). The other tensor components, however, have to vanish for all j. This gives an example where the theorem applies to explicitly discontinuous current in the limit that $\alpha \to 0$. Once again, we remark that the case $\alpha = 0$ is not the case which we need to consider for our regime; we just covered it for completeness.

1.3 Contradiction

Now, we see that the conclusions reached in Step 1 – that the matrix elements are non-zero for all j – contradict Step 2 – that the matrix elements have to vanish for $j > \frac{1}{2}$ or $j > 1$ depending on the setup – in all situations except when $j \leq \frac{1}{2}$ or $j \leq 1$ depending on which situation we are considering. This imposes a limit on the spin of massless particles which are charged under conserved current or tensor.

2 Conclusion and implications

The theorem imposes strict conditions on massless particles. To understand which massless particles *are* allowed, we can rewrite the theorem as [3]

> Massless particles with spin $j > \frac{1}{2}$ ($j > 1$) cannot carry a charge (energy-momentum) induced by a conserved Lorentz-covariant vector (tensor) current.

So, if we want to have a massless particle in our theory that has a large spin, we should ensure it is not charged under a Lorentz-covariant current. This is why gravity is generally regraded as a non-local theory since a particle which mediates it will not be charged under a Lorentz current. There are also detailed discussions in AdS/CFT as to how this limitation for a graviton can be avoided by viewing it as two gluons on the boundary of the dual theory. In any case, this theorem remains invaluable in evaluating the plausibility of any particle theory of gravity, and understanding its proof is the first step to incorporating the result into a theoretical framework.

References

[1] E. Giusti. *Minimal Surfaces and Functions of Bounded Variation*, pages 10 – 20. SpringerScience + BusinessMedia, 1984.

[2] V. Kaplunovky. Saddle point method of asymptotic expansion. https://web2.ph.utexas.edu/~vadim/Classes/2022f/saddle.pdf, 2022.

[3] F. Loebbert. The weinberg-witten theorem on massless particles: an essay. *Annalen der Physik*, 520(9-10):803–829, 2008.

[4] S. Weinberg. *The Quantum Theory of Fields, Volume 1: Foundations*. Cambridge University Press, 2005.

[5] S. Weinberg and E. Witten. Limits on massless particles. *Physics Letters*, 96B(1-2):59–62, October 1980.