Analogue of the Unruh effect in coupled harmonic oscillator and cavity quantum electrodynamics systems.

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This paper represents my work in accordance with University regulations.

/s/ Liza Rozenberg

Abstract

The goal of this paper is to demonstrate that the difficulty of detecting Unruh radiation in Minkowski spacetime can be overcome by using a mathematically analogous system to model it. In this way, we will be able to measure the effect in a different setting while still accounting for the system's most basic features. We will show that the necessary system can be found by considering a pair of coupled optical cavities. We will outline how detection of radiation is to be performed in both systems, and how the two processes are related to each other.

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1 Introduction

The Unruh effect expresses a prediction that in Minkowski spacetime an accelerating observer and an inertial observer will observe empty space differently. The first will detect thermal radiation which will have a finite temperature associated with it, while the latter will not detect any radiation. We call a uniformly accelerating observer a Rindler observer. The stated prediction can be generalized to curved spacetimes in the form of the Hawking effect. Birrell and Davies do this in *Quantum Fields in Curved Space* [1] by using similar techniques to what we will see in the examination of flat spacetime, so this prediction is in fact very general.

The phenomenon of Unruh radiation, and likewise Hawking radiation, is of great interest to us since it has a connection to the study of information in black holes. Unfortunately, the temperature of interest is hard to detect experimentally. To see why, we need to take a look at the expression for the Unruh temperature which we will derive in Section 5. It is given in units where $c = 1$ by $T = \frac{\hbar a}{2\pi k_B}$, where a is the proper acceleration of the detector of radiation. Since $\hbar$ is extremely small, the acceleration required to produce a temperature of just 1 mK is a staggering $2.46 \times 10^{17}\ \frac{\text{m}}{\text{s}^2}$. In order to manage this problem, we look for another system which closely mimics this one in the mathematical sense, and to some extent physical sense, but which is easier to create in a laboratory. We want the essential properties of the system to remain the same, but the temperature to become more experimentally accessible to us. The hope is that given the strong analogy, the predictions we make for one of the systems will hold for the other one as well. One might ask why analogous physical behavior is expected in systems between which we draw a mathematical analogy. We argue for this as follows. To claim otherwise, i.e. that mathematically analogous systems do not yield the same physical behavior, requires one to accept that the mathematical description we give does not capture everything about at least one of the systems. However, in our discussion, we begin with the assumption that we successfully capture all essential features of the systems of interest in their mathematical treatment. As long as we have this assumption, the argument from analogy must follow. Since this is the standpoint from which much of theoretical physics operates, it is not unreasonable to make this assumption in this paper. Further refinements of our knowledge of the systems may arise, of course, but since both systems examined in this paper are indeed very well-studied, our assumptions are well-founded.

The analogy is developed as follows. In the case of the Unruh effect in Minkowski spacetime, we have two wedges of spacetime causally separated from each other. We call this space Rindler space. When we quantize a scalar field in Rindler space, because of the causal separation between the wedges, we have access to two distinct bases of modes to describe any operators in that setting. The two sets of modes are related by a Bogoluibov transformation, as is shown in Section 5, and in more detail in [8]. The same kind of transformation arises in the system made up of two coupled harmonic oscillators. This transformation relates raising and lowering operators from two settings: those annihilating the ground state of the Hamiltonian *with* the coupling term, and those annihilating the ground state of the Hamiltonian *without* the coupling term. The similarity captured by the Bogoluibov transformations is what allows us to expect a similar behavior in both the Rindler and coupled oscillator cases.

In the case of the Rindler observer, we restrict their attention to just one of the wedges (since they simply do not have access to the other one) and perform a measurement of the vacuum as defined by one of the two sets of modes. We use a device which is capable of counting photons and detecting whether a state is excited. As we shall verify in Section 5, we obtain a thermal distribution of particles. Analogously, in the case of the coupled optical cavities, we obtain a temperature by also restricting the measurement to one of the two cavities, which is mathematically modeled by tracing over the Hilbert space of the other subsystem. In this way, we trace over the degrees of freedom inaccessible to us. In both scenarios, the notion of temperature arises as a result of focusing on only one of subsystems.

Now that we have sketched the aspects of the two systems which contain the necessary analogies, we can study them in more detail. We begin by examining the perhaps more familiar of the two settings: the optical cavity and the emergence of harmonic oscillator behavior in it. We will also use the term “cavity QED” to refer to this system.

2 Cavity QED

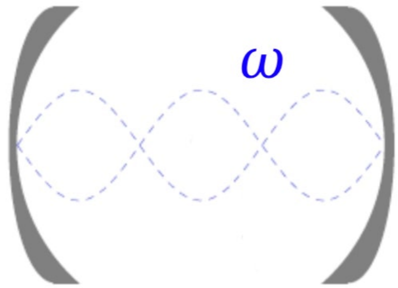

Figure 1: Optical cavity with reflective walls. ω is the characteristic frequency of the system.

Our first step in setting up the more experimentally accessible system is to establish how we can physically model a harmonic oscillator. This can be accomplished by using photons trapped inside an optical cavity. The setup is pictured in figure 1, adapted from [5]. The two sides of the cavity are mirrors, so that a standing electromagnetic wave can form. We shall see in this section that harmonic oscillator behavior arises when we quantize the electromagnetic fields in the cavity and promote all dynamical quantities to operators. As it stands, this is not sufficient to model the Unruh effect, so we will go one step further and add another cavity such that the two interact. Since photons in each cavity have harmonic oscillator behavior, this will give the desired coupled harmonic oscillator system.

2.1 Quantizing the fields

We now perform the quantization of the electromagnetic field in a cavity, which we take without loss of generality to be one-dimensional. The reason we are not losing much information by only considering one dimension is that the discussion which follows readily generalizes to the full three dimensions through linearity and homogeneity of the equations involved via the use of the separation of variables method. The full derivation without appealing to this implicit argument can be found in [4].

We take a one-dimensional cavity of width L in the $\hat{z}$ direction. We first need to pick the polarization of the electric field in the cavity. Since the electric field cannot be polarized in the same direction as that in which the wave associated with it propagates, without loss of generality let us choose it to be polarized in the $\hat{x}$ direction. Solving the wave equation for the electric field obtained from the Maxwell equations, we get (using V to denote the volume of the cavity):

$$\frac{\partial^2 E_x}{\partial z^2} = \frac{1}{c^2}\frac{\partial E_x}{\partial t} \tag{1}$$

$$\therefore \quad E_x(z,t) = \sqrt{\frac{2\omega}{V\epsilon_0}}q(t)\sin(kz) \tag{2}$$

Then, we can use Ampere's law to obtain a similar expression for the magnetic field:

$$\nabla \times \mathbf{B} = \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t} \tag{3}$$

$$\therefore \quad B_y(z,t) = \sqrt{\frac{2\mu_0}{V}}\,\dot{q}(t)\cos(kz) \tag{4}$$

Imposing the boundary conditions at $z = 0$ and $z = L$ which require the electromagnetic field to vanish, we obtain that $k = \frac{n\pi}{L},\ n \in \mathbb{Z}_{\geq 1}$.

From these expressions, we obtain the Hamiltonian

$$\begin{aligned} H &= \frac{1}{2}\int dV\left(\epsilon_0\, E_x^2(z,t) + \frac{B_y^2(z,t)}{\mu_0}\right) \\ &= \frac{1}{2}\left(\dot{q}^2(t) + \omega^2 q^2(t)\right) \end{aligned} \tag{5}$$

This is exactly the Hamiltonian of a simple harmonic oscillator. We can in the usual way promote all dynamical quantities to operators, where we now write momentum as $p(t) = \dot{q}(t)$. Raising and lowering operators can be defined using these promoted quantities as

$$\begin{aligned} \hat{a}(t) &= \frac{1}{\sqrt{2\hbar\omega}}\left(\omega\hat{q}(t) + i\hat{p}(t)\right) \\ \hat{a}^\dagger(t) &= \frac{1}{\sqrt{2\hbar\omega}}\left(\omega\hat{q}(t) - i\hat{p}(t)\right) \end{aligned} \tag{6}$$

These are exactly the operators we know from studying the quantum mechanical harmonic oscillator system. Thus, by performing the quantization, we have shown that in this setting photons behave like harmonic oscillators. We can attribute this behavior to the fact that the boundary conditions at the edges of the cavity impose a countably infinite discrete spectrum of allowed frequencies, and so force photons into a regime with few excitation modes.

In order to use this to study the Unruh effect, we now need to introduce coupling. This is done by adding another cavity such that the two cavities interact. In Section 5 when we examine Rindler space, we will see that the formalism we are about to explore is repeated: the fact that two causally separated wedges of spacetime are permeated by the same scalar field will result in similar mathematical relations.

The picture we want to have in mind when we think of two coupled cavities is shown in figure 2, reproduced from [10].

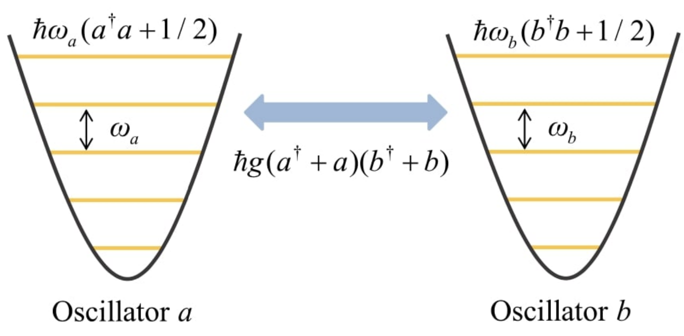

Figure 2: Interacting cavities with indicated energy spacing. In our case, the associated frequencies ω_a and ω_b are the same frequency ω.

Each of the cavities is individually described by its own harmonic oscillator Hamiltonian with a characteristic frequency, which in our case is the same for both. Once we introduce coupling, the interaction is modeled by adding to the Hamiltonian the term $\hbar g(a^\dagger + a)(b^\dagger + b)$ which results in terms involving products of operators associated with different oscillators. This will alter the evolution of the system, as we will now show in Section 2.2.

2.2 Modeling coupling with a quantum beam splitter

We want to better understand the Hamiltonian resulting from introducing coupling in figure 2. With this in mind, consider the Hamiltonian (7), where the parameter α quantifies the extent to which the two cavities interact.

$$H_{\text{osc}} = \hbar\omega(a^\dagger a + \frac{1}{2}) + \hbar\omega(b^\dagger b + \frac{1}{2}) + m\alpha^2(a + a^\dagger)(b + b^\dagger) \tag{7}$$

This is a slight modification of the Hamiltonian in figure 2. The mass parameter m does not actually have to represent a physical mass, but can instead contribute to quantizing how strong coupling between the two cavities is. (In figure 2, g was used instead of $m\alpha^2$.) This is to account for the fact that photons and electromagnetic fields do not have mass, but a Hamiltonian of the form H_{osc} can still be applied to study harmonic oscillator behavior of these massless objects. However, to make the model easier to understand, we can at first think of m as a physical parameter. This will be the approach taken in Section 3.

We can gain a better understanding of the physical picture by interpreting the terms in H_{osc} as shown in figure 3. In this way, we can think of the barrier separating the two cavities as a beam splitter.

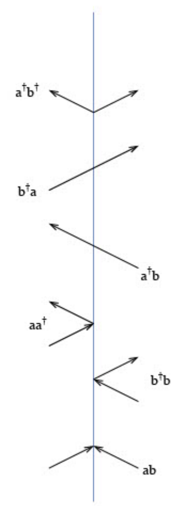

Figure 3: Hamiltonian H_{osc} interpreted as a quantum beam splitter.

The two sets of operators correspond to the two sides, or optical cavities. Mixed terms correspond to allowing the two sides to interact and so for photons to pass across the "barrier". Especially notable are the terms ab and $a^\dagger b^\dagger$, which would not have been present in the classical limit of a beam splitter because they do not conserve energy in the classical sense. For example, the term $a^\dagger b$ represents a photon passing from the left cavity to the right one. Similar interpretations hold for the other terms, bearing in mind that an annihilation operator removes a photon, while a raising operator creates one.

The purpose of highlighting the beam splitter interpretation of (7) is to gain a better understanding of the connection we are making between coupled oscillators and optical cavities. The beam splitter is one way of establishing coupling between optical cavities on either side of the barrier - from figure 3 we see the clear interpretation of the various terms. Since the beam splitter is a realizable optical setup, it directly gives us a viable experimental model for coupling. Although to establish how exactly coupling is to be achieved in a laboratory is not the purpose of this paper, it is useful to have this idea complement our theoretical picture. As we shall see in Section 3.2, a temperature can be associated to this system. We will show that experimentally, this temperature can be measured by counting photons which end up at a detector positioned on the specified side of the division.

3 The coupled harmonic oscillator

Introducing the coupling Hamiltonian is the first step to understanding how temperature arises in this system. To progress further, we now need to formalize our mathematical understanding of the ground state of the coupled harmonic oscillator. We will connect these examinations to the physical picture by showing that by only considering what happens to one of the two coupled subsystems, we obtain a physical notion of temperature. Our task is most easily accomplished by first going back to the more classical picture of viewing coupling as connecting two masses with springs. Once we understand this simpler picture, we can then promote all dynamical quantities to quantum mechanical operators and return to the more abstract operator picture we started with, which can be directly applied to the study of photons.

3.1 The Hamiltonian

We take two identical particles of mass m. The coordinates x_1 and x_2 correspond to the positions of the two oscillators. Note that we will slightly modify the form of the Hamiltonian (7) to make the mathematical treatment easier, but the essential features are kept the same.

Let us examine the components of the Hamiltonian carefully. First, we have the usual uncoupled harmonic oscillator component which acts on the particles symmetrically:

$$H_0 = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}m\omega^2(x_1^2 + x_2^2) \tag{8}$$

The angular frequency parameter ω indicates the strength of this potential. Secondly, we have the coupling term between the particles. It is written as

$$H_{\text{coup}} = \frac{1}{2}m\alpha^2(x_1 - x_2)^2, \tag{9}$$

The parameter α indicates the strength of the coupling.

In full, the coupled system is described by the sum of these terms:

$$H_{\text{osc}} = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}m\omega^2(x_1^2 + x_2^2) + \frac{1}{2}m\alpha^2(x_1 - x_2)^2, \tag{10}$$

The subscripts $i = 1, 2$ label position and momentum operators associated with the first and second particles respectively. We set $m = 1$ from now on to make notation simpler.

The momentum and position operators have the usual raising and lowering operators associated with them, which we write for indices $j = 1, 2$ as

$$a_j = \frac{1}{\sqrt{2\omega}}(p_j - i\omega x_j), \quad a_j^\dagger = \frac{1}{\sqrt{2\omega}}(p_j + i\omega x_j). \tag{11}$$

These satisfy the usual bosonic commutation relations

$$[a_j, a_k^\dagger] = \delta_{jk}, \quad [a_j, a_k] = [a_j^\dagger, a_k^\dagger] = 0. \tag{12}$$

We are interested in finding the ground state of the system described by the Hamiltonian in (10). The first step in doing this is switching to a coordinate system where the Hamiltonian does not have the cross term. In other words, we want to diagonalize it. Consider the following coordinates x_+, x_- which we call "normal":

$$\begin{aligned} x_+ &= \frac{x_1 + x_2}{\sqrt{2}}, \qquad x_- = \frac{x_1 - x_2}{\sqrt{2}} \\ p_+ &= \frac{p_1 + p_2}{\sqrt{2}}, \qquad p_- = \frac{p_1 - p_2}{\sqrt{2}}. \end{aligned} \tag{13}$$

The transformed Hamiltonian is given by:

$$H_{\text{osc}} = \frac{p_+^2}{2m} + \frac{p_-^2}{2m} + \frac{m}{2}\omega^2 x_+^2 + \frac{m}{2}(\omega^2 + 2\alpha^2)x_-^2 \tag{14}$$

We see that this is just a direct sum of two independent (non-interacting) quantum harmonic oscillators, with their respective frequencies

$$\omega_1 = \omega \quad \text{and} \quad \omega_2 = \sqrt{\omega^2 + 2\alpha^2} \tag{15}$$

The new coordinates give us new raising and lowering operators in the standard way for two harmonic oscillators:

$$a_+ = \frac{1}{\sqrt{\omega_1}}(p_+ - i\omega_1 x_+), \quad a_- = \frac{1}{\sqrt{\omega_2}}(p_- - i\omega_2 x_-) \tag{16}$$

This allows us to write the Hamiltonian as

$$H_{\text{osc}} = \hbar\omega_1 a_+^\dagger a_+ + \hbar\omega_2 a_-^\dagger a_- \tag{17}$$

We can interpret these two eigenmodes as two normal modes associated with this system, as depicted in figure 4 reproduced from [3]. The frequency ω_1 corresponds to the mode where the two masses oscillate in the same direction at this frequency, while ω_2 corresponds to the mode where they oscillate in opposite directions each with this frequency.

We can one final time rewrite H_{osc} to make it easier to find the ground state. This is done by defining new raising and lowering operators:

$$\begin{aligned} b_1 = \frac{a_1 - \xi a_2^\dagger}{\sqrt{1-\xi^2}}, \quad b_2 = \frac{a_2 - \xi a_1^\dagger}{\sqrt{1-\xi^2}}, \\ \text{where} \quad \xi = \frac{\omega_1 - \omega_2}{\omega_1 + \omega_2}. \end{aligned} \tag{18}$$

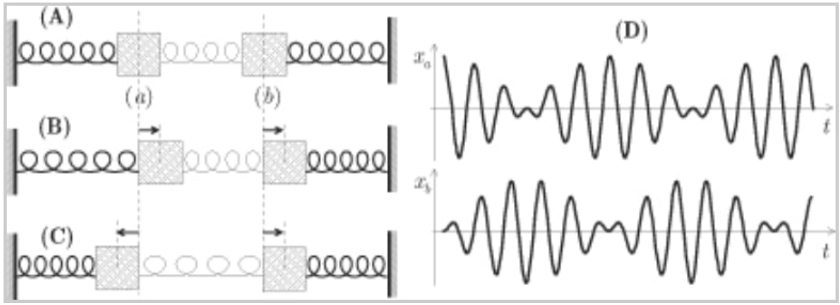

Figure 4: Eigenmodes of two coupled harmonic oscillators: (B) symmetrical and (C) asymmetrical. (D) shows time evolution of the system when only oscillator (a) is excited, where x_a and x_b represent displacement from equilibrium (the dotted lines) of oscillators a and b respectively.

We verify that they satisfy the usual bosonic commutation relations, using the ones we already know for $a_j, a_j^\dagger$:

$$\begin{aligned}
[b_j, b_k^\dagger] &= \frac{1}{1-\xi^2}[a_j - \xi a_k^\dagger,\ a_k^\dagger - \xi a_j] = \frac{1}{1-\xi^2}\Big(\underbrace{[a_j, a_k^\dagger]}_{\delta_{jk}} - \xi[a_j, a_j] - \xi[a_k^\dagger, a_k] + \xi^2 \underbrace{[a_k^\dagger, a_j]}_{-\delta jk}\Big) \\
&= \delta_{jk} \\
[b_j, b_k] &= \frac{1}{1-\xi^2}[a_j - \xi a_k^\dagger,\ a_k - \xi a_j^\dagger] = \frac{1}{1-\xi^2}\Big([a_j, a_k] - \xi[a_j, a_j^\dagger] - \xi[a_k^\dagger, a_k] + \xi^2[a_k^\dagger, a_j^\dagger]\Big) \\
&= 0 \\
\therefore \quad [b_j^\dagger, b_k^\dagger] &= -[b_j, b_k]^\dagger = 0
\end{aligned} \tag{19}$$

Also, we redefine frequencies to get

$$\omega_+ = \frac{1-\xi}{1+\xi}, \qquad \omega_- = \frac{1+\xi}{1-\xi} \tag{20}$$

Using these operators and frequencies, we can rewrite our Hamiltonian as

$$H_{\text{osc}} = \hbar\omega_+ b_1^\dagger b_1 + \hbar\omega_- b_2^\dagger b_2 \tag{21}$$

We shall for later purposes call the modes associated with these new operators and frequencies the "coupling modes", contrasted with the number modes characterized by the operators in (11) and the frequency ω.

Now, based on the new variables, we guess an ansatz for the ground state of H_{osc}:

$$|\Psi\rangle = A \sum_n \xi^n \, |n, n\rangle \tag{22}$$

Acting with the annihilation operator b_1 on this state, we obtain 0:

$$
\begin{aligned}
b_1 |\Psi\rangle &= \frac{A}{\sqrt{1-\xi^2}} \sum_n \xi^n \left(a_1 - \xi a_2^\dagger\right) |n,n\rangle \\
&= \frac{A}{\sqrt{1-\xi^2}} \left(\sum_{n=1}^{\infty} \xi^n \sqrt{n}\, |n-1,n\rangle - \sum_{n=0}^{\infty} \xi^{n+1} \sqrt{n+1}\, |n,n+1\rangle \right) \\
&= \frac{A}{\sqrt{1-\xi^2}} \left(\sum_{n=1}^{\infty} \xi^n \sqrt{n}\, |n-1,n\rangle - \underbrace{\sum_{n=1}^{\infty} \xi^n \sqrt{n}\, |n-1,n\rangle}_{\text{relabel } n+1 \to n} \right) = 0.
\end{aligned}
\tag{23}
$$

Due to the similarity of their expressions, an analogous calculation follows for $b_2 |\Psi\rangle = 0$. Hence, the wavefunction $|\Psi\rangle$ is annihilated by H_{osc}. By uniqueness of the ground state, this must be the ground state for this system. We need only work out the normalization constant. This can be done most efficiently by writing the density matrix for this state. We will use the property that for density matrices their trace equals 1 to obtain the explicit form for A. We will also obtain a useful connection to thermal properties of the system.

3.2 Computing the temperature of the system

Let us first highlight the usefulness of the density matrix approach. The formalism will allow us to later conveniently talk about restricting attention to only one half of the coupled harmonic oscillator system, i.e. one of the cavities. This is done by performing the mathematical process of tracing over the Hilbert space of one of the subsystems. The idea is to trace out the degrees of freedom of the whole system which we do not have access to. We shall do this once we compute the density matrix for the full system. Although the state of the full system is pure as we shall justify below, the state which is described by the reduced density matrix is mixed. We can interpret this as coming from the fact that when we trace over the Hilbert space of one half of the coupled harmonic oscillator system, we do not have precise information about that part of the system anymore, and so the resulting state becomes a statistical ensemble.

Tentatively, we write the density matrix of the full system in the ground state of the coupled Hamiltonian as

$$
\rho = |\Psi\rangle\langle\Psi| \tag{24}
$$

This is the right form for the density matrix if the state is pure. Purity of the state can indeed be checked by verifying directly using the expression (22) that $\rho^2 = \rho$.

A density matrix should always have trace equal to 1. To compute A, we impose this condition and compute:

$$
\begin{aligned}
1 &= \mathrm{Tr}(\rho) \\
A^2 \sum_n \xi^{2n} &= \frac{A^2}{1-\xi^2} \\
\therefore \quad A &= \sqrt{1-\xi^2}
\end{aligned}
\tag{25}
$$

Now that we have the normalization factor, we can write down the density matrix describing the ground state of the Hamiltonian H_{osc}:

$$\rho = (1-\xi^2)\sum_{n,m}\xi^n\xi^m|n,n\rangle\langle m,m| \tag{26}$$

When we restrict our attention to one of the cavities, i.e. one half of the coupled harmonic oscillator system, we obtain a reduced density matrix which is also a thermal density matrix for that subsystem. Without loss of generality, due to the symmetry of the system, let us focus on oscillator 1. To do this, we trace over the Hilbert space of oscillator 2 to obtain the reduced density matrix

$$\begin{aligned}\rho^1 &= \mathrm{Tr}_2(\rho)\\ &= (1-\xi^2)\sum_n \xi^{2n}|n\rangle\langle n|\end{aligned} \tag{27}$$

It is worth noting that the tracing out of one of the subsystems is intended to be useful for cases when we do not have access to that part of the system. This mimics the way in Rindler space, an observer in one wedge of the Rindler space is causally separated from the other wedge, and so does not have (full) information about the other wedge. Acknowledging the missing information is what is being represented by the process of tracing.

The density matrix obtained in (27) can in particular be taken as a thermal density matrix. From this form, we would expect there to emerge a physical temperature. To see what this temperature would have to be, we can match the expression obtained in (27) to the general form of a thermal density matrix which describes statistical mixtures presented as a thermal distribution. With $\beta = \frac{1}{k_B T}$, the matrix should take the form

$$\begin{aligned}\rho^1_{n,n} &= \frac{e^{-\beta\hbar\omega(n+\frac{1}{2})}}{\sum_m e^{-\beta\hbar\omega(m+\frac{1}{2})}} = (e^{\frac{\beta\hbar\omega}{2}} - e^{-\frac{\beta\hbar\omega}{2}})e^{-\beta\hbar\omega(n+\frac{1}{2})}\\ &= 2\sinh{(\frac{\beta\hbar\omega}{2})}e^{-\beta\hbar\omega(n+\frac{1}{2})}\end{aligned} \tag{28}$$

Comparing (27) and (28), we can deduce

$$\begin{aligned}\xi^{2n} - \xi^{2n+2} &= e^{-\beta\hbar\omega n} - e^{-\beta\hbar\omega-\beta\hbar\omega n}\\ \therefore \quad \xi &= e^{-\frac{\beta\hbar\omega}{2}}\end{aligned} \tag{29}$$

Taking log of both sides, we obtain

$$2\log\xi = -\frac{1}{k_B T}\hbar\omega \tag{30}$$

$$\therefore \quad T = -\frac{\hbar\omega}{2k_B\log\xi} = \frac{\hbar\omega}{2k_B\log\frac{1}{\xi}} \tag{31}$$

This is the physical temperature we expect to find when we carry out the necessary measurements, which we will discuss in detail in Section 4. What is particularly interesting about this is that we associate the temperature with the entire system, when in fact measurement of only one of the subsystems is what produces the temperature. This is the only way it makes sense to talk about the temperature of the system if we want to be able to measure it using the methods we outline. The reason for this will become clear once we introduce the final piece in our analogy: a two-level system (atom) which serves as a detector.

Going back to the theoretical picture, we see that by tracing over the part of the system only related to oscillator 2, we have obtained a mixed state. This can be shown by checking that $\rho^2 \neq \rho$ for the reduced state in (27). Additionally, the fact that after doing the tracing the resulting state is mixed tells us that the initial pure state was an entangled state. Thus, it makes sense that by tracing, we have lost some information about the full system.

The above discussion suggests that entanglement is significant in the examination of temperature in this system. With this in mind, we note that a useful concept related to thermal properties of the system is entanglement entropy. It is defined as $S = -\operatorname{Tr}(\rho \log \rho)$, where ρ is the reduced density matrix of a subsystem, in our case oscillator 1. Entropy of entanglement can be understood as a way to quantify entanglement in the full system. For the mixed state (27) of one half of the coupled harmonic oscillator system, we compute

$$S = -\log\left(1 - \xi^2\right) - \log \xi \, \frac{2\xi^2}{1 - \xi^2} \tag{32}$$

The fact that $S \neq 0$ for the reduced density matrix proves that the state associated with this matrix is not pure. This could of course also be checked by computing that $\rho^2 \neq \rho$ for the reduced state. In addition, what we obtain in (32) is the same as the usual thermal entropy corresponding to this mode. This can be attributed to the fact that the reduced density matrix is diagonal. As we shall see later, in the aspect of entanglement entropy the coupled harmonic oscillator system differs somewhat from the Rindler space picture. In the latter system, we will also be able to talk about entropy of entanglement, but unlike in this situation, the expression will be a sum that does not converge to a finite value. Nevertheless, the fact that the entropy of entanglement is non-zero, as we shall see, will be enough to indicate that there is entanglement present between the two Rindler wedges. We will return to these ideas in Section 5.

4 Measuring the photon number

We already know from Section 2 that coupled optical cavities is what is used to model the coupled harmonic oscillator system described in detail above. Now that we understand the theoretical origin of the temperature which arises in this system, we need to discuss how to experimentally measure it. We shall do this by adding another element to the pair of optical cavities: a two-level system represented by an atom. It can be in an excited or in a ground state, depending on whether it has absorbed or released a quantum of energy. The theory needed to describe this new development is captured by the Jaynes-Cummings model, as we shall see in this section.

4.1 Introducing a two-level system

We introduce a two-level system which is characterized by two parameters: ω_0 and Ω. ω_0 is the atomic transition frequency dictating the spacing between the ground state and the excited state, while Ω determines how the coupled harmonic oscillators (optical cavities) interact with the two-level system. The setup is illustrated in figure 5 adapted from [5]. We assume the two-level system only interacts with the coupled system via cavity 2. The reason we need to implement this assumption is because if we directly coupled the atom to both cavites at the same time, we would be faced with needing to model a system which has the atom on one side half of the time, and on the other side the other half of the time. This is not a reasonable experimental setup, so we need to pick a side which we place the atom on and so with which to establish coupling.

We can view placing the atom in one of the cavites as instantaneously decoupling the oscillators and "measuring" one of them. By this action, we mean allowing the two-level system to interact with the cavity. By symmetry, we have without loss of generality picked cavity 2 to be the one which is being interacted

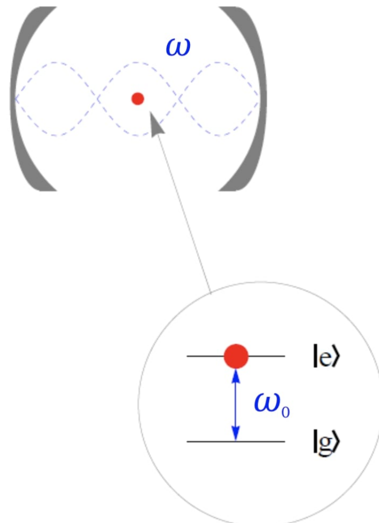

Figure 5: An optical cavity with an atom (two-level system).

with. Despite this simplification, we still have the parameter ξ incorporated into the state which the system starts in. This will later help us calculate the temperature from experimental results.

First, we need to define additional operators associated with the two-level system, which is described by either being in its ground or excited state, given by $|g\rangle$ and $|e\rangle$ respectively. The required operators are:

$$\begin{aligned} \hat{\sigma}_+ &= |e\rangle\langle g| \\ \hat{\sigma}_- &= |g\rangle\langle e| \\ \hat{\sigma}_z &= |e\rangle\langle e| - |g\rangle\langle g| \end{aligned} \tag{33}$$

The modified Hamiltonian which now includes interaction with the two-level system takes the form

$$H_{\text{JCM}}^{\text{coup}} = \underbrace{\hbar\omega_1(a_+^\dagger a_+ + \frac{1}{2}) + \hbar\omega_2(a_-^\dagger a_- + \frac{1}{2})}_{H_{\text{osc}}} + \overbrace{\frac{1}{2}\hbar\omega_0\sigma_z}^{H_{\text{atom}}} + \underbrace{\hbar\Omega(\sigma_+ a_2 + \sigma_- a_2^\dagger)}_{H_{\text{int}}} \tag{34}$$

This Hamiltonian has the familiar H_{osc} term as well as two additional terms. H_{atom} models behavior of the atom as would be the case if no cavities were present. H_{int} models the interaction between the two-level system and cavity 2. Frequencies ω_1 and ω_2 are as in Section 3.

We begin in the ground state of the coupled harmonic oscillator and the ground state of the two-level system. This state is written as

$$\begin{aligned} |\Psi\rangle &= \sum_n c_n |n, n\rangle |g\rangle \\ c_n &= \sqrt{(1-\xi^2)}\xi^n, \quad \xi = \frac{\omega_1 - \omega_2}{\omega_1 + \omega_2} \end{aligned} \tag{35}$$

We want to know what happens when we let this state evolve in time under $H_{\text{JCM}}^{\text{coup}}$. We know already that H_{osc} annihilates $|\Psi\rangle$. The rest of the Hamiltonian we can treat by considering its effect on each 2×2

subspace (except for the 1×1 one with just $|0,0\rangle|g\rangle$) consisting of basis vectors $|n,n\rangle|g\rangle,\ |n,n-1\rangle|e\rangle$, where $|e\rangle$ is the excited state of the atom.

Fixing n, we look at the nth subspace. We will call the part of $|\Psi\rangle$ which corresponds to this subspace $|\psi_n(t)\rangle$. We denote $|i\rangle = |n,n\rangle|g\rangle$ and $|f\rangle = |n,n-1\rangle|e\rangle$. Then with general coefficients we can write

$$|\psi_n(t)\rangle = c_i(t)|i\rangle + c_f(t)|f\rangle \tag{36}$$

We now solve the time dependent part of the Schrödinger equation with H_{int} as the Hamiltonian that produces the time dependence. This is sufficient since H_{atom} merely flips the sign of $|g\rangle$ and multiplies it by a constant, thus merely providing an overall phase that can be disregarded. The phase can be disregarded because we are interested in this process to find the probability that the state will evolve into something involving $|e\rangle$, which will be unaffected by a constant phase after taking absolute value later on. Then,

$$i\hbar\frac{\partial|\psi_n\rangle}{\partial t} = H_{\text{int}}|\psi_n\rangle \tag{37}$$

Since $|i\rangle$ and $|f\rangle$ are independent, we match their coefficients once we apply the above equation and get

$$\begin{aligned}\dot{c}_i &= -i\Omega\sqrt{n}c_f\\ \dot{c}_f &= -i\Omega\sqrt{n}c_i\end{aligned} \tag{38}$$

Combining these with initial conditions from $|0\rangle_{coupled}$ given by $c_i(0) = c_n$ and $c_f(0) = 0$, we get

$$\begin{aligned}&\ddot{c}_i + n\Omega c_i = 0\\ \therefore\quad &c_i = c_n\cos{(\Omega\sqrt{n}t)},\end{aligned} \tag{39}$$

and similarly

$$\therefore\quad c_f = -ic_n\sin{(\Omega\sqrt{n}t)}. \tag{40}$$

We finally write

$$|\psi(t)\rangle = c_0|0,0\rangle|g\rangle + \sum_{n=1}^{\infty} c_n[\cos{(\Omega\sqrt{n}t)}|n,n\rangle|g\rangle - i\sin{(\Omega\sqrt{n}t)}|n,n-1\rangle|e\rangle] \tag{41}$$

Then, we can just read off the probability of the atom being in the excited state as $\sum_{n=1}^{\infty} c_n^2\sin^2{(\Omega\sqrt{n}t)}$. Using $c_n = (\sqrt{1-\xi^2})\xi^n$, we can write the probability of the two-level system going into the excited state as

$$\text{P}_{\text{exc}} = (1-\xi^2)\sum_{n=1}^{\infty}\xi^{2n}\sin^2{(\Omega\sqrt{n}t)} \tag{42}$$

4.2 Measuring the temperature

Now that we have obtained an expression for the probability of measuring the atom in an excited state, we can use this information to suggest a way to extract the experimental value of the temperature of the subsystem associated with cavity 2. (Note that now cavity 2 plays the role of oscillator 1 in Section 3.) We can read off the probability values for measuring the atom in an excited state and n photons in cavity 2 as

$$\mathrm{P}^{\,n}_{\mathrm{exc}} = (1-\xi^2)\ \xi^{2(n+1)} \sin^2\left(\Omega\sqrt{n+1}\,t\right) \tag{43}$$

With all of this information, we can perform many measurements of the two-level system, each time preparing the full system in the same way to be in state (35). With the experimental results of the fraction of all trials which result in n photons and an excited atom, we obtain a value for $\mathrm{P}^{\,n}_{\mathrm{exc}}$ and n. Using the formula (43), we obtain a value for ξ. As we derived in (31), it only depends on $\hbar$, k_B, ω, and T. The first three of these quantities we know already, so we can work out T in terms of known constants.

Alternatively, we do not need to know what Ω is since we can just take the measurement when probability is maximum. This occurs when t is such that $\sin^2\left(\Omega\sqrt{n+1}\,t\right) = 1$, or when $t = \frac{(2m+1)\pi}{2}\frac{1}{\Omega\sqrt{n+1}}$, $m \in \mathbb{Z}_{\geq 0}$. In that case, we can take measurements at one of the times we just noted, using the same setup during each trial. To verify that the two-level system is indeed in the excited state, we can wait for a time of order $\frac{1}{\Omega}$ [7] for an emission of a photon to confirm that the system was indeed in the excited state, and has now transitioned to the ground state. Using the value of the fraction of all measurements that resulted in an excited atom with n photons in cavity 2, we can compute probability and hence temperature using (43).

Using the formalism developed in this section, we can finish explaining why this way of attributing a temperature to the full system is the only way it makes sense to do so given our methods, despite the fact that it seems to only take into account one of the two subsystems. As we saw above, the atom is coupled directly to one of the cavities, and it is not really feasible to couple it to both cavities at the same time. Then, when the atom, understood now as the detector, "makes a measurement", it interacts with only the cavity to which it is coupled. In this setting, there is no way for it to measure what is happening with the other cavity, and in general in any experimental setup we will always have to stay ignorant about some of the degrees of freedom present to gain certainty about one part of the system. This is the scenario in which temperature arises, and it expresses our ignorance about some part of the full system.

5 The Unruh effect

Now that we have developed the tools to study the cavity QED system, we can return to the system which was of interest to us in the first place: a uniformly accelerating observer (detector) in Rindler space. In this section, we will look closely at the mathematical description of this system. We base our discussion on the approach found in [1]. Coupling will arise implicitly between regions of Minkowski spacetime which are causally separated but still interacting by virtue of being permeated by the same scalar field. Throughout this section, we will point out parallels between this system and concepts mentioned in the preceding sections. Note that we will be working in the system of units where the speed of light is $c = 1$, and that we will be using the signature $(+,-,-,-)$ for the Minkowski metric.

Minkowski spacetime has line element given by

$$ds^2 = dt^2 - dx^2 \tag{44}$$

Note that we only write the spatial coordinate as x. This is because in Rindler coordinates, the y and z coordinates remain unchanged by the transformations, so we might as well focus on the one dimensional spatial picture.

The coordinate and wedge structure is depicted in figure 6 reproduced from [1]. We are interested in looking at two of the four wedges into which we divide Minkowski space: wedges R and L. These wedges are referred to as Rindler space, and we denote them collectively by $\mathcal{R}$.

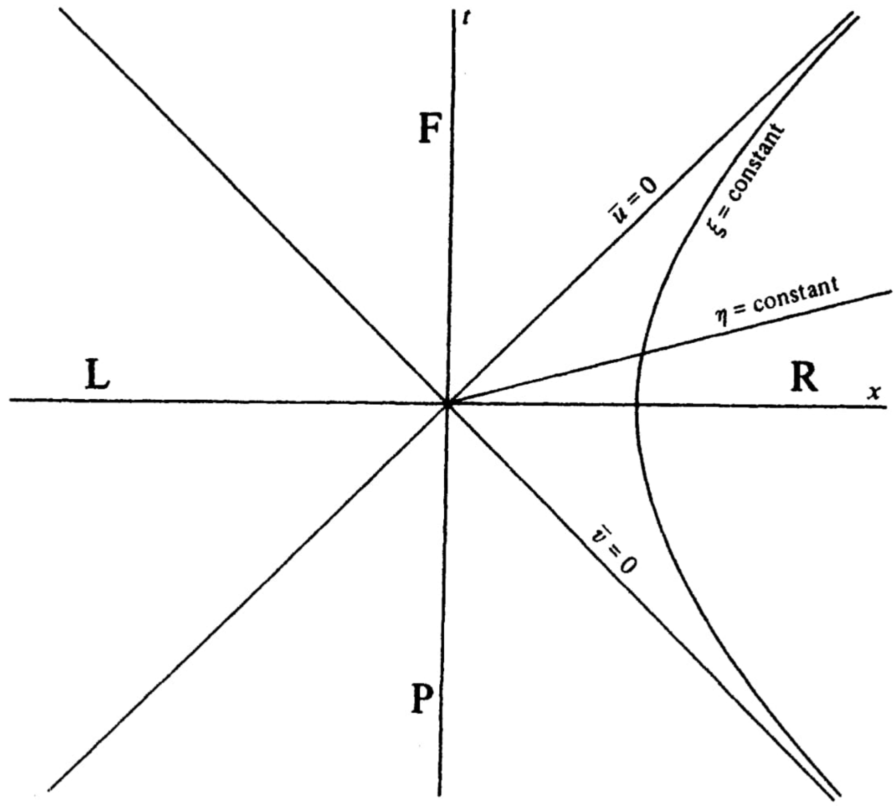

Figure 6: Splitting Rindler space into wedges.

In the right Rindler wedge R where $x > |t|$, we can use the coordinate transformations

$$\begin{aligned} t &= \frac{1}{a} e^{a\xi} \sinh a\eta \\ x &= \frac{1}{a} e^{a\xi} \cosh a\eta \end{aligned} \tag{45}$$

$$\begin{aligned} u = \eta - \xi, \ \ v = \eta + \xi \\ a > 0, \ -\infty < \ \eta, \xi \ < \infty \end{aligned} \tag{46}$$

to rewrite the line element as

$$ds^2 = e^{2a\xi} du dv = e^{2a\xi}(d\eta^2 - d\xi^2) \tag{47}$$

We should establish a few features about the new coordinates: η acts as the time coordinate and ξ as the spatial coordinate. Using $\bar{u} = -\frac{e^{-au}}{a}$, $\bar{v} = \frac{e^{av}}{a}$, we can define event horizons $\bar{u} = 0$ and $\bar{v} = 0$ on the coordinate diagram since Rindler observers approach but do not cross these lines.

Similarly, we can represent the left Rindler wedge L where $x < |t|$ by reversing the signs of (45):

$$t = -\frac{1}{a} e^{a\xi} \sinh a\eta$$
$$x = -\frac{1}{a} e^{a\xi} \cosh a\eta \tag{48}$$

We want to learn about the ground state of the system. To do this, we can take a similar approach to what we did in Section 2: we shall quantize the scalar field ϕ. It has to satisfy the Klein-Gordon equation in this space, so

$$\Box\phi = \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\phi = 0 \tag{49}$$

We can view this relation as an analogy to the scalar fields in an optical cavity satisfying the Maxwell equations - the form of the Klein-Gordon equation is similar to the wave equations derived from Maxwell equations.

Since the transformation from (t, x) to (η, ξ) is conformal, and the Klein-Gordon equation is conformally invariant [1], we directly obtain

$$\left(\frac{\partial^2}{\partial \eta^2} - \frac{\partial^2}{\partial \xi^2}\right)\phi = 0. \tag{50}$$

The solution to this equation is given separately in R and L, and by restricting the support of the distinct parts of the solution to their own regions, we write the complete set of solutions as

$$^{R}u_k = \begin{cases} \frac{1}{\sqrt{4\pi\omega}} e^{ik\xi - i\omega\eta} & \text{in R} \\ 0 & \text{in L} \end{cases} \tag{51}$$

and

$$^{L}u_k = \begin{cases} \frac{1}{\sqrt{4\pi\omega}} e^{ik\xi + i\omega\eta} & \text{in L} \\ 0 & \text{in R} \end{cases} \tag{52}$$

where $\omega = |k| > 0,\ -\infty < k < \infty$. The set of solutions also includes their complex conjugates, but we do not write them out explicitly here since they can be obtained by simply taking the complex conjugate of (49) and (50). These modes satisfy the necessary normalization conditions $(u_i, u_j) = \delta_{i,j},\ (u_i^*, u_j^*) = -\delta_{i,j},$ and $(u_i, u_j^*) = 0$, where $(\ ,\ )$ denotes scalar product.

Now we note that ∂_η is a future-directed Killing field in R, while $\partial_{-\eta}$ is a future-directed Killing field in L [1]. We want the solutions in both wedges to have positive frequency relative to their respective future-directed Killing fields (since we need a notion of positive energy in each wedge). On their support, the modes satisfy

$$\partial_\eta(^{R}u_k) = -i\omega\,(^{R}u_k)$$
$$\partial_{(-\eta)}(^{L}u_k) = -i\omega\,(^{L}u_k) \tag{53}$$

This set of so called Rindler modes can be contrasted with the Minkowski set of modes which are valid solutions of the Klein-Gordon equation on the entire Minkowski space, as opposed to only on the R and L wedges as is the case for the above constructions. The Minkowski modes can simply be written as

$$\bar{u}_k = \frac{1}{\sqrt{4\pi\omega}} e^{ikx - i\omega t} \tag{54}$$

with ω and k as before. The Rindler and the Minkowski modes each define a set of raising and lowering operators. At this point, it is important to note the parallel between these two sets, and the sets we obtained when we were discussing coupling in the previous sections. The Minkowski modes are like the usual photon number modes (of the type $|n, m\rangle$), while the Rindler modes are like the coupling modes of the coupled harmonic oscillator.

Due to completeness and orthogonality of our sets of modes, we can expand the scalar field ϕ in terms of two different sets of raising and lowering operators. The first set is $\{a_k, a_k^\dagger\}$, which correspond to the Minkowski space $\mathcal{M}$ and apply to the entire space. The other set is $\{b_k^{(1)}, b_k^{(1)\dagger}, b_k^{(2)}, b_k^{(2)\dagger}\}$, defined on R and L. The superscript (1) associates an operator to the L region, while (2) associates it to the R region. We note that the Hilbert space is the same in either representation since the inner product is the same, but its interpretation as a Fock space will differ between $\mathcal{M}$ and $\mathcal{R}$.

Using this, we can expand ϕ in two ways:

$$\begin{aligned} \phi &= \sum_{k=-\infty}^{\infty} \left(a_k \bar{u}_k + a_k^\dagger \bar{u}_k^* \right) \\ \text{and} \quad \phi &= \sum_{k=-\infty}^{\infty} \left(b_k^{(1)} \left({}^L u_k\right) + b_k^{(1)\dagger} \left({}^L u_k^*\right) + b_k^{(2)} \left({}^R u_k\right) + b_k^{(2)\dagger} \left({}^R u_k^*\right) \right) \end{aligned} \tag{55}$$

These two expressions are related by a Bogoliubov transformation as outlined in [8].

Similarly, we have two vacuum states $|0_\mathcal{M}\rangle$ and $|0_\mathcal{R}\rangle$ for Minkowski and Rindler spaces respectively. Since these are ground states, they must satisfy

$$\begin{aligned} a_k|0_\mathcal{M}\rangle &= 0 \\ b_k^{(1)}|0_\mathcal{R}\rangle = b_k^{(2)}|0_\mathcal{R}\rangle &= 0 \end{aligned} \tag{56}$$

We must note that these vacuum states are not the same. To see why this is the case, consider the Rindler modes at $t = 0$. They only have support on the half-line, which means it is not possible to express this mode using only positive-frequency plane waves in $\mathcal{M}$ since we need to annihilate the mode in the necessary half of the line. Hence, Rindler annihilation operators which are used to define $|0_\mathcal{R}\rangle$, have to be superpositions of Minkowski creation and annihilation operators, so the same state will not be annihilated by both sets. This shows the vacuums are not the same.

Since $\partial_{\pm\eta}$ are Killing vector fields, we know that a Rindler observer will be static with respect to orbits along $\partial_{\pm\eta}$. This observer will detect no particles in the Rindler vacuum. However, when traveling through Minkowski vacuum, from the above discussion about superpositions of positive and negative frequency modes in the expression for Rindler modes, we expect that our accelerating observer will detect particles in Minkowski vacuum.

To analyze this phenomenon mathematically, we first find a set of modes which share the vacuum state with the Minkowski modes but which have simpler relations to the Rindler modes. We do this by first taking the original Rindler modes, extending them analytically to the entire spacetime, then expressing this extension in terms of the original Rindler modes. At the end of the process, we obtain the properly normalized extended modes:

$$
\begin{aligned}
&\text{Extension of } {}^{R}u_k: \qquad d_k^{(1)} = \frac{1}{\sqrt{2\sinh\left(\frac{\pi\omega}{a}\right)}} \left(e^{\frac{\pi\omega}{2a}} \left({}^{R}u_k\right) + e^{-\frac{\pi\omega}{2a}} \left({}^{L}u_{-k}^{*}\right)\right) \\
&\text{Extension of } {}^{L}u_k: \qquad d_k^{(2)} = \frac{1}{\sqrt{2\sinh\left(\frac{\pi\omega}{a}\right)}} \left(e^{\frac{\pi\omega}{2a}} \left({}^{L}u_k\right) + e^{-\frac{\pi\omega}{2a}} \left({}^{R}u_{-k}^{*}\right)\right)
\end{aligned}
\tag{57}
$$

Denoting by $c_k^{(1,2)}, c_k^{(1,2)\,\dagger}$ the annihilation and creation operators associated with the new modes, we can rewrite the scalar field as

$$
\phi = \sum_{k=-\infty}^{\infty} \left(c_k^{(1)} d_k^{(1)} + c_k^{(1)\dagger} d_k^{(1)*} + c_k^{(2)} d_k^{(2)} + c_k^{(2)\dagger} d_k^{(2)*}\right) \tag{58}
$$

We can obtain the relations between new and old annihilation and creation operators directly from the relations between the new and old sets of modes:

$$
\begin{aligned}
b_k^{(1)} &= \frac{1}{\sqrt{2\sinh\left(\frac{\pi\omega}{a}\right)}} \left(e^{\frac{\pi\omega}{2a}} c_k^{(1)} + e^{-\frac{\pi\omega}{2a}} c_{-k}^{(2)\dagger}\right) \\
b_k^{(2)} &= \frac{1}{\sqrt{2\sinh\left(\frac{\pi\omega}{a}\right)}} \left(e^{\frac{\pi\omega}{2a}} c_k^{(2)} + e^{-\frac{\pi\omega}{2a}} c_{-k}^{(1)\dagger}\right)
\end{aligned}
\tag{59}
$$

These are called Bogoliubov transformations, and they are mathematically analogous to the transformations between the two sets of operators in Section 3 associated with the coupled harmonic oscillators.

The vacuum state for these new modes is the same as the one for the old positive-frequency Minkowski modes. This is because the old Minkowski positive-frequency modes with $k > 0$ satisfy $\bar{u}_k \propto e^{-i\omega(t-x)}$ and are analytic and bounded for complex (t, x) as long as $\mathrm{Im}(t - x) \leq 0$. We can say the same for our new modes $d_k^{(1)}$ as long as we stay in the upper-half complex $(t - x)$ plane.

Similar arguments follow to determine that $h_k^{(2)}$ are analytic and bounded in the lower-half complex $(t+x)$. This leads to the conclusion that the new modes $h_k^{(1,2)}$ may be expressed purely in terms of positive-frequency Minkowski plane-wave modes u_k. This directly implies that the two sets of modes share a vacuum state, hence

$$
c_k^{(1)}|0_{\mathcal{M}}\rangle = c_k^{(2)}|0_{\mathcal{M}}\rangle = 0 \tag{60}
$$

Finally, we take a Rindler observer in R who is uniformly accelerating, hence moving along a constant ξ curve. Since their proper time is proportional to η, we expect that their vacuum corresponds to $|0_{\mathcal{R}}\rangle$. Hence, we expect them to be able to detect particles in $|0_{\mathcal{M}}\rangle$. We carry out the computation:

$$
\begin{aligned}
\langle 0_{\mathcal{M}}|{}^{R}n^{(k)}|0_{\mathcal{M}}\rangle &= \langle 0_{\mathcal{M}}|b_k^{(1)\dagger}\, b^{(1)}|0_{\mathcal{M}}\rangle \\
&= \frac{1}{2\sinh\left(\frac{\pi\omega}{a}\right)} \langle 0_{\mathcal{M}}|e^{-\frac{\pi\omega}{a}}\, c_{-k}^{(1)}\, c_{-k}^{(1)\dagger}|0_{\mathcal{M}}\rangle \\
&= \frac{e^{-\frac{\pi\omega}{a}}}{2\sinh\left(\frac{\pi\omega}{a}\right)} \\
&= \frac{1}{e^{\frac{2\pi\omega}{a}} - 1}
\end{aligned}
\tag{61}
$$

This corresponds to a Planck spectrum with temperature $T = \frac{\hbar a}{2\pi k_B}$. More precisely, this is the temperature that will be measured by a Rindler observer moving along the path $\xi = 0$. For any other constant ξ, the acceleration felt is $\alpha = ae^{-a\xi}$, so the measured temperature is $T = \frac{\hbar\alpha}{2\pi k_B}$ [1]. Detection of radiation in this manner is what we refer to as the Unruh effect.

One final thing we want to confirm is that a notion of coupling is present in this scenario, in direct analogy with the cavity QED system described earlier in the paper. To do this, let us compute the entanglement entropy for the state $|0_{\mathcal{M}}\rangle$. It can be written as a sum of terms

$$S = \sum_{\omega} \left(-\log\left(1-\xi^2\right) - \log\xi \,\frac{2\xi^2}{1-\xi^2} \right) \tag{62}$$

As before, $\xi = e^{-\frac{\beta\hbar\omega}{2}}$. We have to carry out the sum over all possible modes, i.e. over ω, so the sum is infinite. This is because there are infinitely many pairs of points, one in each wedge, that are coupled to each other as harmonic oscillators. This is the result of quantizing a scalar field in Rindler space. In the cavity QED case, we only had two coupled oscillators, which is why our expression was finite.

However, note that each of the terms in (62) is positive since $\xi < 1$. Hence, the entropy of entanglement satisfies $S > 0$, telling us that the full state is indeed entangled. Hence, our analogy holds here also between cavity QED and Rindler space: there is a notion of entanglement present in both systems.

6 Conclusion

We have successfully computed the temperature detected in the vacuum state of $\mathcal{M}$ by a uniformly accelerating observer. Due to the similarity of transformations (18) and (59), mathematically, the interaction between the R and L wedges is analogous to the coupling between the two cavities we studied in the preceding sections. Hence, it makes sense to think of the two cavities as modeling the two wedges, and the temperature observed in the case of cavity QED as analogous to the temperature being observed by a Rindler observer in the context of the Unruh effect.

In Section 5, we pointed out that Rindler modes are like coupling modes in the coupled harmonic oscillator case, and Minkowski modes are like the number modes. When we derived the ground state wavefunction of the coupled system in (22), we saw that the number modes are present, but the coupling modes are not (since (22) is annihilated by the operators b_1 and b_2 from Section 3). This was the reason we obtained a thermal distribution when observing that ground state - we were counting the number modes. In a similar manner, in the discussion in Section 5 we showed that in Minkowski vacuum, the Minkowski modes are not excited while the Rindler modes are. Hence, it makes sense that a thermal particle distribution, and hence a temperature, was obtained here also.

The Rindler space analysis then tells us that detection of particles can be observer dependent. By virtue of the analogy pointed out above, a question arises. What exactly do we call a photon? Is it an excitation of an individual cavity, or is it an excitation of the individual eigenmodes? In other words, are they excitations of number modes (also Minkowski modes) or the coupling modes (also Rindler modes)?

That this question arises should not be surprising. In fact, we come across a similar problem throughout the field of quantum mechanics, since we must always make a choice of basis when writing down a wavefunction. The basis choice determines what constitutes an excited mode. Similarly, when performing an experiment, we must choose what it is we are measuring. For instance, we might tune our detector to detect the emitted frequency ω, or ω_1, ω_2. This choice will determine whether a photon is detected or not. In that sense then, and in line with the design of the theoretical detector in [8], a Minkowski observer is like a detector tuned to detect frequency ω. Hence, what we think of as a photon is determined by what exactly is being measured by the tuned detector.

In this paper, we have outlined a theoretical experiment that we can use to measure the temperature using the appropriately tuned detector, but it remains to be examined closer whether it is reasonable to expect this to be a realizable setup in the setting of cavity QED. To make it realizable, we need to achieve a measurable and hopefully convenient probability for detecting the needed state. Let us consider a suitable value of ξ we can implement. Note that from (18) we know that it is always less than 1. Looking at (43), we can differentiate to find the extremum. The reason this gives the maximum, not the minimum, is because we can make the probability 0 by setting $\xi = 0$, meaning the local extremum has to be a maximum. In this manner, we find that the value of ξ which gives maximum probability is $\xi = \sqrt{\frac{n+1}{n+2}}$.

Let us consider a possible setting. If we want our setup to detect only one photon, we can set $\xi = \sqrt{\frac{2}{3}} \approx 0.816$, and by running the experiment 100 times we obtain ≈ 14 instances of an excited atom with a single photon in cavity 2. Then we use the procedure outlined in Section 4 to compute temperature from these experimental results. If we wanted to detect a different value of n photons, we would increase the number of trials to make the fraction of successes still detectable in an experimental setup. In comparison with the enormous acceleration needed to produce a detectable temperature in the Rindler space scenario, the cavity QED setup is much more accessible. Since we have preserved the essential mathematical and physical aspects of the system, we expect analogous observations to hold for the Rindler space scenario, too.

We shall conclude this paper with an open question that is raised by the discussion above. Why does there exist such a strong similarity in the mathematical description of the cavity QED system and the Rindler space system, while only the latter has an explicit notion of acceleration associated to it? Acceleration of the Rindler observer is directly responsible for the detection of particles in Minkowski vacuum, since as we saw a Minkowski (non-accelerating) observer will not detect any. The hope is that there is a way to find an "effective" notion of acceleration in the cavity QED system, although it must arise in a different way than in the Rindler case. In the latter system, it comes directly from considerations of the metric describing the space [1], but there is little hope of defining a metric for the cavity QED case since there are only two "points" and an atom. Perhaps what needs to be done now is to reverse the approach taken in this paper and instead view cavity QED using the formalism which has been developed for geometrically understanding spacetime.

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